# Urcohomologies and Cohomologies of N-Complexes

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# Urcohomologies and cohomologies of $N$-complexes

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Dedicated to the late Professor Leopoldo Nachbin

(Communicated by Rui Loja Fernandes)

**Abstract.** For a general sequence of objects and morphisms, we construct two $N$-complexes. Then we can define cohomologies $(i,k)$-type of the $N$-complexes not only on a diagonal region but also in the triangular region. We obtain an invariant defined on a general sequence of objects and morphisms. For a short exact sequence of $N$-complexes, we get the associated long exact sequence generalizing the classical long exact sequence. Lastly, several properties of the vanishing cohomologies of $N$-complexes are given.

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**Keywords.** $N$-complex, $N$-exactness, complexifying functor, urcohomology.

## 1. Introduction

The notion of an $N$-complex was introduced in [9], where $N = 2$ is the usual complex. The cohomologies of $N$-complexes have been investigated in [9], [4], [11], [17], [1], [3], [5]. In addition to the cohomologies located on the diagonal as studied previously, we will study the cohomologies in the triangular region (see Fig. 3).

Our motivation to study cohomologies defined for $N$-complexes and also for general sequences (the most general) comes partially from comments made by Kapranov in [9], "It is natural to ask why $\mathrm{d}^2$ and not, say, $\mathrm{d}^3$. Following this mood, we give the natural definition (of an $N$-complex)" for introducing the notion of an $N$-complex. Furthermore, we can ask why $\mathrm{d}^N$ and not $\mathrm{d}^\infty$. Namely, it may be natural to ask whether there exists a stronger invariant defined for any sequence of objects and morphisms which coincides with the classical cohomology for $n = 2$ and also the cohomology for general $n = N$. This ultimate case is the notion of urcohomology. We introduce in this paper two functors from the urcategory consisting of objects and morphisms (without any additional conditions) to the category of $N$-complexes for $N = 2, 3, \dots$.

In order to define a cohomology-like invariant on an arbitrary sequence of objects and morphisms, denoted as $(A^\bullet, f^\bullet)$ where $(f)^N = 0$ need not hold for any $N \geq 2$, a right exact functor $\mathrm{I}^{-N}$ and a left exact functor $\mathrm{K}^N$ are introduced. Those functors are said to be $N$-complexifying functors since $\mathrm{I}^{-N}(A^\bullet)$ and $\mathrm{K}^N(A^\bullet)$ become $N$-complexes for an arbitrary sequence $(A^\bullet, f^\bullet)$. The cohomologies of such complexified objects $\mathrm{I}^{-N}(A^\bullet)$ and $\mathrm{K}^N(A^\bullet)$ are said to be the urcohomologies of $(A^\bullet, f^\bullet)$. We have the self-duality

$$\mathrm{H}^j(\mathrm{K}^N(A^\bullet)) \xrightarrow{\approx} \mathrm{H}^j(\mathrm{I}^{-N}(A^\bullet)).$$

Urcohomologies for the case $N = 2$ are treated in [14], [13] in which such invariants are called precohomologies instead of urcohomologies.

## 2. Cohomology of $(i, k)$-type and urcohomology

Let $\mathscr{A}$ be an abelian category. A sequence $(A^\bullet, \mathrm{d}^\bullet)$ of objects and morphisms of $\mathscr{A}$ is said to be an $N$-complex if

$$(\mathrm{d})^N = \mathrm{d}^{j+N-1} \circ \mathrm{d}^{j+N-2} \circ \cdots \circ \mathrm{d}^j = 0, \qquad j \in \mathbb{Z},$$

where $\mathrm{d}^j : A^j \to A^{j+1}$. Let $\mathsf{Ur}(\mathscr{A})$ be the category of general sequences $(A^\bullet, f^\bullet)$ of objects $A^\bullet$ and morphisms $f^\bullet$ of $\mathscr{A}$ where a morphism $\phi^\bullet$ between objects $(A^\bullet, f^\bullet)$ and $(B^\bullet, g^\bullet)$ of $\mathsf{Ur}(\mathscr{A})$ is a morphism $\phi^j : A^j \to B^j$ of $\mathscr{A}$ satisfying the commutativity $g^j \circ \phi^j = \phi^{j+1} \circ f^j$ for $j \in \mathbb{Z}$ in the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A^j \arrow[r, "f^j"] \arrow[d, "\phi^j"] & A^{j+1} \arrow[r] \arrow[d, "\phi^{j+1}"] & \cdots \\
\cdots \arrow[r] & B^j \arrow[r, "g^j"] & B^{j+1} \arrow[r] & \cdots .
\end{tikzcd}$$

The abelian category $\mathsf{Co}^N(\mathscr{A})$ of $N$-complexes is a full subcategory of $\mathsf{Ur}(\mathscr{A})$.

Based on the following two principles (1) and (2) for an $N$-complex $(A^\bullet, \mathrm{d}^\bullet)$, we can define more cohomological objects in addition to

$$\mathrm{Ker}(\mathrm{d})^k / \mathrm{Im}(\mathrm{d})^i$$

on the diagonal $i + k = N$. Those principles are

$$(\mathrm{d})^N = (\mathrm{d})^{k+i} = (\mathrm{d})^{k+k'+i+i'} = 0, \qquad \text{for } k', i' \geq 0 \tag{1}$$

and

$$\begin{cases} A^j = \mathrm{Ker}(\mathrm{d})^N \hookleftarrow \mathrm{Ker}(\mathrm{d})^{N-1} \hookleftarrow \cdots \hookleftarrow \mathrm{Ker}(\mathrm{d}) \\ A^j = \mathrm{Im}(\mathrm{d})^0 \hookleftarrow \mathrm{Im}(\mathrm{d}) \hookleftarrow \mathrm{Im}(\mathrm{d})^2 \hookleftarrow \cdots \hookleftarrow \mathrm{Im}(\mathrm{d})^{N-1}. \end{cases} \tag{2}$$

That is, we get cohomologies located in the region $i + k \geq N$.

**Definition 2.1.** Let $(C^\bullet, \mathrm{d}^\bullet)$ be an $N$-complex. For any integers $j$ and $(i, k)$ satisfying $i + k \geq N$ and $0 < i, k < N$, the subquotient object of $C^j$

$$ {}_i\mathrm{H}^j_k(C^\bullet) = \mathrm{Ker}(\mathrm{d})^k / \mathrm{Im}(\mathrm{d})^i $$

is said to be the $j$-th cohomology of $(i, k)$-type of $(C^\bullet, \mathrm{d}^\bullet)$.

**Remark 2.2.** For an $N$-complex $(C^\bullet, \mathrm{d}^\bullet)$, we have $\frac{N(N-1)}{2}$ cohomologies in the following triangular region:

$$\begin{array}{lccccccc}
{\scriptstyle N)} & 0 \hookrightarrow \mathrm{Ker}(\mathrm{d})^1 & \hookrightarrow \mathrm{Ker}(\mathrm{d})^2 \hookrightarrow & \cdots & & \hookrightarrow \mathrm{Ker}(\mathrm{d})^{N-1} & \hookrightarrow & C^j \\
 & \downarrow & & & & \downarrow & & \downarrow \\
 & {}_{N-1}\mathrm{H}^j_1 & \hookrightarrow \cdots & \cdots & & \hookrightarrow {}_{N-1}\mathrm{H}^j_{N-1} & \hookrightarrow & C^j/\mathrm{Im}(\mathrm{d})^{N-1} \\
 & & & & & \downarrow & & \downarrow \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
 & & \downarrow & \downarrow & & & & \\
{\scriptstyle i+1)} & & {}_{i+1}\mathrm{H}^j_{k-1} & \hookrightarrow {}_{i+1}\mathrm{H}^j_k \cdots & & & & \\
 & & & \downarrow & & & & \\
{\scriptstyle i)} & & & {}_i\mathrm{H}^j_k & \hookrightarrow & {}_i\mathrm{H}^j_{k+1} \cdots & & \vdots \\
\vdots & & & \ddots & & \vdots & & \\
 & & & & & & & \downarrow \\
 & & & & & & & C^j/\mathrm{Im}(\mathrm{d})^2 \\
 & & & & & \downarrow & & \downarrow \\
 & & & & & {}_1\mathrm{H}^j_{N-1} & \hookrightarrow & C^j/\mathrm{Im}(\mathrm{d})^1 \\
 & & & & & & & \downarrow \\
 & & & & & & & 0. \\
 & \cdots & & k & & \cdots & & N
\end{array} \tag{3}$$

The cohomologies in the triangular region in (3) are obtained from the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}(\mathrm{d})^{N-1} \arrow[rr] \arrow[d, hook] \arrow[dr] \arrow[drr] \arrow[ddrr] & & \mathrm{Ker}(\mathrm{d})^{1} \arrow[r] \arrow[d, hook] & {}_{N-1}\mathrm{H}^{j}_{1} \arrow[r] \arrow[d, "\tilde{\imath}"] & 0 \\
0 \arrow[dr] & \vdots \arrow[d, hook] & {} & \vdots \arrow[d, hook] & \vdots \arrow[d, "\tilde{\imath}"] & \\
0 \arrow[r] & \mathrm{Im}(\mathrm{d})^{3} \arrow[rr] \arrow[d, hook] \arrow[ddrr] & & \mathrm{Ker}(\mathrm{d})^{N-3} \arrow[r] \arrow[d, hook] & {}_{3}\mathrm{H}^{j}_{N-3} \arrow[r] \arrow[d, "\tilde{\imath}"] & 0 \\
0 \arrow[r] & \mathrm{Im}(\mathrm{d})^{2} \arrow[rr] \arrow[d, hook] \arrow[drr] & & \mathrm{Ker}(\mathrm{d})^{N-2} \arrow[r] \arrow[d, hook] & {}_{2}\mathrm{H}^{j}_{N-2} \arrow[r] \arrow[d, "\tilde{\imath}"] & 0 \\
0 \arrow[r] & \mathrm{Im}(\mathrm{d})^{1} \arrow[rr] \arrow[d, hook] & & \mathrm{Ker}(\mathrm{d})^{N-1} \arrow[r] \arrow[d, hook] \arrow[dr] \arrow[ddr] & {}_{1}\mathrm{H}^{j}_{N-1} \arrow[r] & 0 \\
& \mathrm{Im}(\mathrm{d})^{0} \arrow[r, equal] & C^{j} \arrow[r, equal] & \mathrm{Ker}(\mathrm{d})^{N} & {}_{2}\mathrm{H}^{j}_{N-1} \arrow[u, two heads, "\tilde{1}"'] \arrow[dr] & \\
& & & & {}_{3}\mathrm{H}^{j}_{N-1} \arrow[u, two heads, "\tilde{1}"'] \arrow[dr] & 0 \\
& & & & & 0
\end{tikzcd}
\tag{4}
$$

where $\tilde{1}$ and $\tilde{\imath}$ in (4) are induced by an identity morphism and a monomorphism respectively. For $\mathrm{Im}(\mathrm{d})^i$ in (4), there are $(i+1)$ morphisms into $\mathrm{Ker}(\mathrm{d})^k$ where $k = N-1, N-(i-1), \dots, N-1, N$. Namely, we get the cohomologies of $(i, N-i), \dots, (i, N)$-types, which appear horizontally at the $i$-th level of (3). For $\mathrm{Ker}(\mathrm{d})^k$, we get the cohomologies appearing vertically in (3) at the $k$-th level. Cohomologies treated in [9], [4], [11], [17], [3], [5] are located only on the diagonal $i + k = N$ in (3).

We define a useful functor which makes an $N$-complex into a 2-complex.

**Definition 2.3.** For an $N$-complex $(C^\bullet, \mathrm{d}^\bullet) \in \mathrm{Ob}(\mathsf{Co}^N(\mathscr{A}))$, we define $\delta^j_{(k)} : C^j \to C^{j+k}$ as

$$\delta^j_{(k)} = (\mathrm{d})^k : C^j \to C^{j+k}.$$

And we consider a sequence

$$\cdots \xrightarrow{\delta^{j-i-i'}_{(i')}} C^{j-i} \xrightarrow{\delta^{j-i}_{(i)}} C^j \xrightarrow{\delta^j_{(k)}} C^{j+k} \xrightarrow{\delta^{j+k}_{(k')}} \cdots$$

where $i+i' \geq N, i+k \geq N, k+k' \geq N, \ldots$. Then the sequence above is a complex $(C^\bullet, \delta^\bullet_{(*)}) \in \mathsf{Co}(\mathscr{A})$. We call this assignment from $\mathsf{Co}^N(\mathscr{A})$ to $\mathsf{Co}(\mathscr{A})$ the second bundling functor $\mathsf{b} : \mathsf{Co}^N(\mathscr{A}) \rightsquigarrow \mathsf{Co}(\mathscr{A})$. Moreover in the case that $i+i' = N, i+k = N, k+k' = N, \ldots$, such a functor is called the standard bundling functor.

At the later half of this section, we will define the first bundling functor which makes not only an $N$-complex, but an arbitrary sequence into a 2-complex (see Definition 2.13).

**Proposition 2.4.** *Let*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0 \tag{5}$$

*be a short exact sequence of $N$-complexes. Then for any $j \in \mathbb{Z}$ and any $(i,k)$ satisfying $i+k \geq N$, there is the induced long exact sequence;*

$$\begin{aligned}
\cdots &\longrightarrow {}_{i'}\mathrm{H}_i^{j-i}(A^\bullet) \longrightarrow {}_{i'}\mathrm{H}_i^{j-i}(B^\bullet) \longrightarrow {}_{i'}\mathrm{H}_i^{j-i}(C^\bullet) \\
&\xrightarrow{\partial^{j-i}} {}_i\mathrm{H}_k^j(A^\bullet) \longrightarrow {}_i\mathrm{H}_k^j(B^\bullet) \longrightarrow {}_i\mathrm{H}_k^j(C^\bullet) \\
&\xrightarrow{\partial^{j}} {}_k\mathrm{H}_{k'}^{j+k}(A^\bullet) \longrightarrow {}_k\mathrm{H}_{k'}^{j+k}(B^\bullet) \longrightarrow {}_k\mathrm{H}_{k'}^{j+k}(C^\bullet) \\
&\xrightarrow{\partial^{j+k}} \cdots,
\end{aligned}$$

*where $i+i' \geq N, i+k \geq N, k+k' \geq N, \ldots$.*

*Proof.* By the second bundling functor in Definition 2.3, the above assertion becomes the usual long exact sequence of cohomologies induced by the short exact sequence of 2-complexes. That is, by the second bundling functor $\mathsf{b}$, the short exact sequence (5) of $N$-complexes becomes the short exact sequence

$$0 \to \mathsf{b}A^\bullet \to \mathsf{b}B^\bullet \to \mathsf{b}C^\bullet \to 0$$

of 2-complexes. Since $\mathsf{b}$ is an exact functor, we get the long exact sequence corresponding to $i = i' = k = k' = \cdots = 1$. $\square$

A similar result to Proposition 2.4 can be found in [17].

**Remark 2.5.** The notion of homotopy for $N$-complexes appears in [11], [17]. That is, morphisms $\phi^\bullet$ and $\psi^\bullet : A^\bullet \to B^\bullet$ of $N$-complexes are said to be homotopic when there exists a morphism $s^\bullet : A^\bullet \to B^\bullet[-(N-1)]$ satisfying

$$\phi - \psi = \sum_{i=0}^{N-1} (\mathrm{d}_B)^{N-1-i} \circ s \circ (\mathrm{d}_A)^i$$

for any $j \in \mathbb{Z}$. As proved in loc. cit., two homotopic morphisms induce the same map on cohomologies in the diagonal region. However, we remark that this is not the case in the whole triangular region.

Next we define functors which construct an $N$-complex from an arbitrary sequence of objects and morphisms.

**Definition 2.6.** We define functors $\mathrm{I}^{-N}$ and $\mathrm{K}^N$ from the category $\mathsf{Ur}(\mathscr{A})$ of sequences of objects and morphisms of an abelian category $\mathscr{A}$ to the category $\mathsf{Co}^N(\mathscr{A})$ of $N$-complexes as follows. For $(A^\bullet, f^\bullet) \in \mathsf{Ob}(\mathsf{Ur}(\mathscr{A}))$,

$$\mathrm{I}^{-N}(A^\bullet) = A^\bullet/\mathrm{Im}(f)^N = \left(A^j/\mathrm{Im}(f)^N\right)_{j\in\mathbb{Z}}$$

and

$$\mathrm{K}^N(A^\bullet) = \mathrm{Ker}(f)^N = \left(\mathrm{Ker}(f)^N\right)_{j\in\mathbb{Z}}.$$

Then we have the following lemmas.

**Lemma 2.7.** *For $(A^\bullet, f^\bullet) \in \mathsf{Ob}(\mathsf{Ur}(\mathscr{A}))$, $\mathrm{I}^{-N}(A^\bullet)$ and $\mathrm{K}^N(A^\bullet)$ are $N$-complexes.*

**Lemma 2.8.** *Functors $\mathrm{I}^{-N}$ and $\mathrm{K}^N$ are right exact and left exact functors from $\mathsf{Ur}(\mathscr{A})$ to $\mathsf{Co}^N(\mathscr{A})$, respectively.*

Proofs for these lemmas are trivial.

**Remark 2.9.** Let $F : \mathscr{A} \to \mathscr{B}$ be a functor of abelian categories $\mathscr{A}$ and $\mathscr{B}$. Then the following functors are induced:

$$\mathsf{Ur}(F) : \mathsf{Ur}(\mathscr{A}) \rightsquigarrow \mathsf{Ur}(\mathscr{B})$$

and

$$\mathsf{Co}^N(F) : \mathsf{Co}^N(\mathscr{A}) \rightsquigarrow \mathsf{Co}^N(\mathscr{B}).$$

**Example 2.10.** The following is typical of the case $N = 2$. Let $(A^\bullet, f^\bullet)$ be the sequence of free $\mathbb{Z}$-modules:

$$A^\bullet : \cdots \xrightarrow{4} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{4} \cdots.$$

Then we have

$$\mathrm{I}^{-2}(A^\bullet) : \cdots \xrightarrow{\bar{4}} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\bar{2}} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\bar{4}} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\bar{2}} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\bar{4}} \cdots.$$

And clearly $\mathrm{K}^2(A^\bullet) = 0$, since $\mathrm{Ker}(f)^2 = 0$. It is easy to see that $\mathrm{I}^{-2}(A^\bullet)$ is also trivial. See Remark 2.12 (2) below. See also [6], §1.

**Theorem 2.11.** *Let $(A^\bullet, f^\bullet)$ be an object of $\mathsf{Ur}(\mathscr{A})$. Then for the $N$-complexes $\mathrm{I}^{-N}(A^\bullet)$ and $\mathrm{K}^N(A^\bullet)$, there exists an isomorphism of cohomologies*

$$ {}_i\mathrm{H}_k^j(\mathrm{K}^N(A^\bullet)) \xrightarrow{\approx} {}_i\mathrm{H}_k^j(\mathrm{I}^{-N}(A^\bullet)). $$

*Such an isomorphic object in $\mathscr{A}$ is said to be the $j$-th urcohomology of $(i,k)$-type evaluated at $A^\bullet$ denoted as ${}_i\mathrm{h}_k^j(A^\bullet)$.*

**Remark 2.12.**

(1) In [13] and [14], the terminology precohomology is used instead of urcohomology for $N = 2$. We consider that "ur-" meaning "primitive" is better suited for this invariant. This is partly because, as provided by Deligne, an urcohomology fails to be half-exact (see Remark 2.15).

(2) It is well known that the urcohomology is trivial for the fake complex

$$ \cdots \xrightarrow{\varphi} S^{(n)} \xrightarrow{\psi} S^{(n)} \xrightarrow{\varphi} S^{(n)} \xrightarrow{\psi} \cdots $$

of periodic 2 in the matrix factorization of a regular local ring $S$ in [6], [19].

(3) A summarizing commutative diagram can be given as

$$ \begin{tikzcd} & \mathsf{Ur}(\mathscr{A}) \arrow[dl, "\mathrm{I}^{-N}"'] \arrow[dr, "\mathrm{K}^N"] \arrow[dd, "{}_i\mathrm{h}_k^j"] & \\ \mathsf{Co}^N(\mathscr{A}) \arrow[dr, "{}_i\mathrm{H}_k^j"'] & & \mathsf{Co}^N(\mathscr{A}) \arrow[dl, "{}_i\mathrm{H}_k^j"] \\ & \mathscr{A}. & \end{tikzcd} $$

To prove Theorem 2.11, we define the following complexifying functors.

**Definition 2.13.** For a sequence $(A^\bullet, f^\bullet) \in \mathsf{Ob}(\mathsf{Ur}(\mathscr{A}))$, we define morphisms in the same way as Definition 2.3:

$$ \delta_{(k)}^j = (f)^k : A^j \to A^{j+k}, $$
$$ \delta_{(N-k)}^{j+k} = (f)^{N-k} : A^{j+k} \to A^{j+N}. $$

We have $\delta_{(N-k)}^{j+k} \circ \delta_{(k)}^j = (f)^{N-k} \circ (f)^k = (f)^N : A^j \to A^{j+N}$. Then the $N$-complex $\mathrm{K}^N(A^\bullet) = \mathrm{Ker}(f)^N$ (resp. $\mathrm{I}^{-N}(A^\bullet) = A^\bullet/\mathrm{Im}(f)^N$) becomes the usual complex (i.e., 2-complex) $\mathrm{Ker}(\delta)^2$ (resp. $A^\bullet/\mathrm{Im}(\delta)^2$). We denote the assignments from $\mathsf{Ur}(\mathscr{A})$ to $\mathsf{Co}^N(\mathscr{A})$ by

$$ {}'\mathrm{K}^2(A) : (A^\bullet, f^\bullet) \rightsquigarrow (\mathrm{Ker}(\delta)^2), $$
$$ {}'\mathrm{I}^{-2}(A) : (A^\bullet, f^\bullet) \rightsquigarrow (A^\bullet/\mathrm{Im}(\delta)^2), $$

respectively. The complexifying functors $'\mathrm{K}^2$ and $'\mathrm{I}^{-2}$ are said to be the first bundling functors from $\mathsf{Ur}(\mathscr{A})$ to $\mathsf{Co}^2(\mathscr{A}) = \mathsf{Co}(\mathscr{A})$.

*Proof of Theorem* 2.11. For $(A^\bullet, f^\bullet) \in \mathrm{Ob}(\mathsf{Ur}(\mathscr{A}))$, we can regard ${}_i\mathrm{H}_k^j(\mathrm{K}^N(A^\bullet))$ and ${}_i\mathrm{H}_k^j(\mathrm{I}^{-N}(A^\bullet))$ as ${}_1\mathrm{H}_1^j('\mathrm{K}^2(A^\bullet)) = \mathrm{H}^j('\mathrm{K}^2(A^\bullet))$ and ${}_1\mathrm{H}_1^j('\mathrm{I}^{-2}(A^\bullet)) = \mathrm{H}^j('\mathrm{I}^{-2}(A^\bullet))$ by the first bundling functors in Definition 2.13. Then the assertion becomes the self-duality theorem in [13], p. 67. □

**Remark 2.14.** Theorem 2.11 can be proved without the bundling functors as follows. We will give the proof using the exact embedding theorem of an abelian category $\mathscr{A}$ whose objects form a set (i.e., $\mathscr{A}$ is small) into the category of abelian groups (see [16] for details). We also use $\bar{x}$ and $[x]$ in what will follow for the element in a subquotient object (like a cohomological object) and in a quotient object respectively. For the morphism of the composition

$$0 \to \mathrm{Ker}(f)^N \xrightarrow{\iota} A^j \xrightarrow{\pi} A^j/\mathrm{Im}(f)^N \to 0,$$

the cohomologies taken in $\mathsf{Co}^N(\mathscr{A})$ induce the morphism

$${}_i\mathrm{H}_k^j(\mathrm{K}^N(A^\bullet)) \xrightarrow{\overline{\pi\circ\iota}} {}_i\mathrm{H}_k^j(\mathrm{I}^{-N}(A^\bullet))$$

in $\mathscr{A}$. We will prove that $\overline{\pi\circ\iota}$ is an isomorphism.

Suppose that $\overline{\pi\circ\iota}(\bar{x}) = \bar{0}$ in ${}_i\mathrm{H}_k^j(\mathrm{I}^{-N}(A^\bullet))$, where $\bar{x} \in {}_i\mathrm{H}_k^j(\mathrm{K}^N(A^\bullet))$. That is, for $x \in \mathrm{Ker}(f|_K)^k$, we have in $A^j/\mathrm{Im}(f)^N$

$$\pi(\iota(x)) = [x] \in \mathrm{Im}([f])^i, \qquad [x] \in A^j/\mathrm{Im}(f)^N,$$

where $([f])^i : A^{j-i}/\mathrm{Im}(f)^N \to A^j/\mathrm{Im}(f)^N$ in $\mathsf{Co}^N(\mathscr{A})$. Therefore, there exists $[y] \in A^{j-i}/\mathrm{Im}(f)^N$ satisfying $([f])^i([y]) = [x]$, and we have the equality $[(f)^i(y)] = [x]$ in $A^j/\mathrm{Im}(f)^N$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathrm{Ker}(f)^N \arrow[r, "(f|_K)^i"] \arrow[d, "\iota"] & \mathrm{Ker}(f)^N \arrow[r, "(f|_K)^k"] \arrow[d, "\iota"] & \mathrm{Ker}(f)^N \arrow[r] \arrow[d, "\iota"] & \cdots \\
\cdots \arrow[r] & A^{j-i} \arrow[r, "(f)^i"] \arrow[d, "\pi"] & A^j \arrow[r, "(f)^k"] \arrow[d, "\pi"] & A^{j+k} \arrow[r] \arrow[d, "\pi"] & \cdots \\
\cdots \arrow[r] & A^{j-i}/\mathrm{Im}(f)^N \arrow[r, "{[f]^i}"] \arrow[d] & A^j/\mathrm{Im}(f)^N \arrow[r, "{[f]^k}"] \arrow[d] & A^{j+k}/\mathrm{Im}(f)^N \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0. &
\end{tikzcd} \tag{6}$$

Then there are $y$ and $y'$ in $A^{j-i}$ satisfying $x = (f)^i(y) + (f)^i(y')$ since

$$x - (f)^i(y) \in \operatorname{Im}(f)^N \subset \operatorname{Im}(f)^i.$$

We need to show that $y + y' \in \operatorname{Ker}(f)^N \subset A^{j-i}$. For $i + k = N$, compute $([f]^k \circ [f]^i)(\pi(y + y'))$ as

$$[(f)]^{k+i}([y + y']) = [(f)^{k+i}(y + y')] = \pi((f)^{k+i}(y + y')) = [0]$$

in $A^{i+k}/\operatorname{Im}(f)^N$ in (6). Namely we have

$$(f)^N(y + y') = (f)^{k+i}(y + y') \in \operatorname{Ker}(f)^N = \operatorname{Im}\iota = \operatorname{Ker}\pi.$$

**Remark 2.15.** The half-exactness of urcohomologies as claimed in [13] does not hold. The following counter-example is given by P. Deligne. For the exact sequence in $\mathsf{Ur}(\mathscr{A})$:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "1"] \arrow[u] & \mathbb{Z}\oplus 0 \arrow[r] \arrow[u] & 0 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "1"', "-1"] \arrow[u, "1"'] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "1", "1"'] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & \mathbb{Z}\oplus 0 \arrow[r, "1"] \arrow[u, "1"'] & \mathbb{Z} \arrow[r] \arrow[u, "1"] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

the induced urcohomology sequence becomes $0 \to \mathbb{Z} \to 0$.

**Remark 2.16.** The counter-example given in [13] indicating the fact that a urcohomology sequence is not an exact connected sequence and the counter-example in Remark 2.15 are due to the only left exactness and right exactness of $\mathrm{K}^N$ and $\mathrm{I}^{-N}$, respectively.

## 3. The vanishing of $(i, k)$-type cohomologies

In Section 2, we introduced the new cohomological objects for $N$-complexes, which are located in the triangular region. In this section, we investigate the relation between the vanishing of $(i,k)$-type cohomologies in the triangular region and an $N$-exactness of $N$-complex. The definition of an $N$-exactness is introduced by Kapranov [9].

**Definition 3.1.** Let $A^{\bullet}$ be an $N$-complex. Then we say that $A^{\bullet}$ is $N$-exact at $j$ if all the cohomologies in the diagonal region at $j$ vanish. That is, ${}_{i}\mathrm{H}_{k}^{j}(A^{\bullet}) = 0$ for any $(i,k)$ satisfying $i+k=N$ and $0<i,k<N$. And we also say that $A^{\bullet}$ is $N$-exact if it is $N$-exact for all $j \in \mathbb{Z}$.

**Lemma 3.2.** *If an $N$-complex $A^{\bullet}$ satisfies ${}_{N-i}\mathrm{H}_{i+1}^{j}(A^{\bullet}) = 0$ for any $1 \leq i \leq N-2$, then $A^{\bullet}$ is $(2-)$exact at $j$, that is,* $\mathrm{Im}(\mathrm{d}) = \mathrm{Ker}(\mathrm{d})$.

*Proof.* From the diagram (3), we get that

$$\begin{array}{ccc} {}_{N-i}\mathrm{H}_{i}^{j}(A^{\bullet}) & \hookrightarrow & {}_{N-i}\mathrm{H}_{i+1}^{j}(A^{\bullet}) \\ & & \downarrow \\ & & {}_{N-i-1}\mathrm{H}_{i+1}^{j}(A^{\bullet}) \end{array}$$

for any $1 \leq i \leq N-2$. By the hypothesis, we have ${}_{N-i}\mathrm{H}_{i+1}^{j}(A^{\bullet}) = 0$. Combined with the diagram above, this implies that

$${}_{N-i}\mathrm{H}_{i}^{j}(A^{\bullet}) = {}_{N-i}\mathrm{H}_{i+1}^{j}(A^{\bullet}) = {}_{N-i-1}\mathrm{H}_{i+1}^{j}(A^{\bullet}) = 0.$$

Then, we have the following equality for any $1 \leq i \leq N-2$:

$$\mathrm{Ker}(\mathrm{d})^{i} = \mathrm{Im}(\mathrm{d})^{N-i} = \mathrm{Ker}(\mathrm{d})^{i+1} = \mathrm{Im}(\mathrm{d})^{N-i-1}.$$

Hence we have

$$\mathrm{Ker}(\mathrm{d}) = \mathrm{Ker}(\mathrm{d})^{2} = \cdots = \mathrm{Ker}(\mathrm{d})^{(N-2)+1} = \mathrm{Im}(\mathrm{d})^{N-(N-2)-1} = \mathrm{Im}(\mathrm{d}),$$

completing the proof. □

As a corollary, we have the following result.

**Corollary 3.3.** *If an $N$-complex $A^{\bullet}$ satisfies ${}_{i}\mathrm{H}_{k}^{j}(A^{\bullet}) = 0$ for any $j \in \mathbb{Z}$ and $(i,k)$ satisfying $i+k \geq N$ and $0<i,k<N$, then $A^{\bullet}$ becomes a $(2-)$exact complex.*

*Proof.* The assertion holds by Lemma 3.2. □

In the proof of Lemma 3.2, we show that the vanishing condition of ${}_{N-i}\mathrm{H}_{i+1}^{j}(A^{\bullet})$ induces ${}_{N-i}\mathrm{H}_{i}^{j}(A^{\bullet}) = 0$ and ${}_{N-i-1}\mathrm{H}_{i+1}^{j}(A^{\bullet}) = 0$. Using this argument, we have the following.

**Proposition 3.4.** *An $N$-complex $A^\bullet$ is a $(2-)$exact complex at $j$ if ${}_{N-1}\mathrm{H}^j_{N-1}(A^\bullet)$ vanishes.*

From Proposition 3.4, we expect that some vanishing condition of a $(i,k)$-type cohomology induces the $M$-exactness for some $M < N$. Namely, we have the following theorem generalizing Proposition 3.4.

**Theorem 3.5.** *Let $A^\bullet$ be an $N$-complex and let $h$ be an integer satisfying $0 < h \leq N-2$. If one of the $(i,k)$-type cohomologies ${}_i\mathrm{H}^j_k(A^\bullet)$ where $i+k = N+h$ vanishes for any $j \in \mathbb{Z}$, then $A^\bullet$ is an $(N-h)$-exact complex.*

*Proof.* Let $h$ be an integer satisfying $0 < h \leq N-2$. Suppose that $A^\bullet$ is an $N$-complex satisfying ${}_i\mathrm{H}^j_k(A^\bullet) = 0$ for any $j$ and $i+k = N+h$. From the proof of Lemma 3.2, we have

$$ {}_{i'}\mathrm{H}^j_{k'}(A^\bullet) = 0 $$

for any $i' \leq i$ and $k' \leq k$ satisfying $N \leq i'+k' \leq N+h$. Then, we have the equality

$$ \mathrm{Im}(\mathrm{d})^{N-k} = \mathrm{Ker}(\mathrm{d})^{N-i} \tag{7} $$

for any $j$. We note that $(N-k)+(N-i) = N-h$. We will show that for an $N$-complex $A^\bullet$ satisfying (7) for any $j$, $A^\bullet$ is an $(N-h)$-complex. To show this, we need the following trivial lemma.

**Lemma.** *If an $N$-complex $A^\bullet$ satisfies $\mathrm{Im}(\mathrm{d})^i \subseteq \mathrm{Ker}(\mathrm{d})^k$ at any $j \in \mathbb{Z}$, where $i+k = M < N$, then $A^\bullet$ is an $M$-complex.*

(*Proof of Lemma*). Let $(i',k')$ be integers satisfying $i'+k' = M$, and $(i',k') \neq (i,k)$. First, we assume that $i' < i$ and set that $l = i - i'$. For any $x \in \mathrm{Im}(\mathrm{d})^{i'}$, there exists $y \in A^{j-i'}$ to satisfy $(\mathrm{d})^{i'}(y) = x$. Then,

$$ (\mathrm{d})^{k'}(x) = (\mathrm{d})^{k'+i'}(y) = (\mathrm{d})^{k'-l} \circ (\mathrm{d})^{i'+l}(y) = (\mathrm{d})^k \circ (\mathrm{d})^i(y) = 0. $$

The same argument is valid in the case $i' > i$.

To complete the proof, it is enough to show the $(N-h)$-exactness of $A^\bullet$. However, since ${}_{N-k}\mathrm{H}^j_{N-i}(A^\bullet) = 0$ for all $j$ and by [9], Proposition 1.5, the proof follows. □

In general, the converse of Lemma 3.2, Proposition 3.4 and Theorem 3.5 do not hold. However, if we assume the $N$-exactness of an $N$-complex, the converse is also true. That is, we have the following proposition.

**Proposition 3.6.** *Let $A^\bullet$ be an $N$-complex. We assume that $A^\bullet$ is $N$-exact at $j$. If $A^\bullet$ is $(N-h)$-exact at $j$, then all of the $(i,k)$-type cohomologies ${}_i\mathrm{H}_k^j(A^\bullet)$ at $j$ where $i+k=N+h$ vanish.*

*Proof.* Let $(i,k)$ be positive integers satisfying $i+k=N+h$, and set $k'=N-i$ and $i'=N-k$. Then we have $i'+k'=(N-k)+(N-i')=2N-(N+h)=N-h$. Since $A^\bullet$ is $N$-exact and $(N-h)$-exact at $j$, we have

$${}_i\mathrm{H}_{k'}^j(A^\bullet)={}_{i'}\mathrm{H}_k^j(A^\bullet)={}_{i'}\mathrm{H}_{k'}^j(A^\bullet)=0.$$

Then, we have the following equality

$$\mathrm{Im}(\mathrm{d})^i=\mathrm{Ker}(\mathrm{d})^{k'}=\mathrm{Im}(\mathrm{d})^{i'}=\mathrm{Ker}(\mathrm{d})^k.$$

Therefore, we complete the proof. □

**Example 3.7.** We consider the following fake complex of free $\mathbb{Z}$-modules of periodic 3:

$$A^\bullet:\cdots\xrightarrow{c}A^{j-1}\xrightarrow{a}A^j\xrightarrow{b}A^{j+1}\xrightarrow{c}A^{j+2}\xrightarrow{a}\cdots,$$

where $a$, $b$, $c$ are non zero elements in $\mathbb{Z}$ and $A^j=\mathbb{Z}$ for all $j$. Then we have a 3-complex

$$\mathrm{I}^{-3}(A^\bullet):\cdots\xrightarrow{\bar c}\bar A^{j-1}\xrightarrow{\bar a}\bar A^j\xrightarrow{\bar b}\bar A^{j+1}\xrightarrow{\bar c}\bar A^{j+2}\xrightarrow{\bar a}\cdots,$$

where $\bar A^j=\mathbb{Z}/(abc)$. We note that $\mathrm{I}^{-3}(A^\bullet)$ is exact as a 3-complex. Then we have the $(2,2)$-type cohomologies as

$${}_2\mathrm{H}_2^j(\mathrm{I}^{-3}(A^\bullet))=\mathrm{Ker}(\overline{bc}:\bar A^j\to\bar A^{j+2})/\mathrm{Im}(\overline{ac}:\bar A^{j-2}\to\bar A^j).$$

It is easy to see that ${}_2\mathrm{H}_2^j(\mathrm{I}^{-3}(A^\bullet))=0$ if and only if $c\equiv 1 \bmod bc$. Therefore, $\mathrm{I}^{-3}(A^\bullet)$ is $(2-)$exact at $j$ if and only if $c\equiv 1$ mod $bc$ by Theorem 3.5 and Proposition 3.6.

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